

# The Distance Coloring of Graphs

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**Abstract** In this paper we present some upper bounds for the distance chromatic number of a graph in terms of maximum degree subject to the conditions on minimum degree or girth or connectivity, or in terms of the spectral radius of the adjacency matrix of the graph.

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## 1 Introduction

Let  $G = (V(G), E(G))$  be a graph. A *vertex  $k$ -coloring* of  $G$  is a mapping from  $V(G)$  to  $\{1, 2, \dots, k\}$  such that any two adjacent vertices are mapped to different integers. The smallest integer  $k$  for which a  $k$ -coloring exists is called the *chromatic number* of  $G$ , denoted by  $\chi(G)$ . The  *$d$ th power of the graph  $G$* , denoted by  $G^d$ , is a graph on the same vertex set as  $G$  such that two vertices are adjacent in  $G^d$  if and only if their distance in  $G$  is at most  $d$ . The  *$d$ -distance  $k$ -coloring*, also called *distance  $(d, k)$ -coloring*, is a  $k$ -coloring of the graph  $G^d$  (or equivalently, any two vertices within distance  $d$  in  $G$  receive different colors). The  *$d$ -distance chromatic number* of  $G$  is exactly the chromatic number of  $G^d$ , denoted by  $\chi_d(G)$ . Clearly  $\chi(G) = \chi_1(G) \leq \chi_d(G) = \chi(G^d)$ .

The distance coloring was introduced by Florica Kramer and Horst Kramer [10, 11], and a recent survey on this topic was also given by them; see [12] for more details. The  $d$ -distance coloring of graphs has a good application in the frequency assignment problem (radio channel assignment). Graph coloring formalizes this problem well when the constraint is that a pair of transceivers within distance  $d$  cannot use the same channel due to interference; see [13]. The first reference to appear on coloring squares of planar graphs was by Wegner [19]. He posed the following conjecture.

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CONJECTURE 1.1 (Wegner [19]) *Let  $G$  be a planar graph. Then*

$$\chi_2(G) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Some work has been done on the case  $\Delta = 3$ , as listed in [9, Problem 2.18]. Thomassen [18] proved the conjecture holds in this case. For  $\Delta \geq 4$ , the conjecture is still open. Many upper bounds on  $\chi_2(G)$  for planar graphs in terms of  $\Delta$  have been obtained in the last about two decades. The asymptotically best known upper bound so far has been found by Molloy and Salavatipour [14]:  $\chi_2(G) \leq \lceil \frac{5}{3}\Delta \rceil + 78$ . Havet et al. [4] proved that  $\chi_2(G) \leq \frac{3}{2}\Delta + o(1)$  when  $\Delta \rightarrow \infty$ . For a general graph  $G$  of order  $n$ , Huang and Lih [5] obtained two Nordhaus-Guadagni type relations:  $n + 1 \leq \chi_2(G) + \chi_2(\bar{G}) \leq 2n$ ,  $n \leq \chi_2(G)\chi_2(\bar{G}) \leq n^2$ . The left equalities hold when  $G = K_n$  or  $G = O_n$ ; and a graph is constructed to state the sharpness of the upper bounds though they seem trivial.

In this paper we discuss the upper bound of  $\chi_d(G)$  for a general  $d \geq 2$ . These bounds are investigated in two aspects: one is to use the maximum degree subject to the conditions such as minimum degree, girth, vertex connectivity, the other is to use the spectral radius of the adjacency matrix of  $G$ . We review some results below. It was noted in Skupień [17] that the well-known Brooks' theorem can provide the following upper bound:

$$\chi_d(G) \leq 1 + \Delta(G^d) \leq 1 + \Delta \sum_{i=1}^d (\Delta - 1)^{i-1} = 1 + \Delta \frac{(\Delta - 1)^d - 1}{\Delta - 2}. \quad (1.1)$$

For a planar graph  $G$ , Jendrol and Skupien [8] improved it as  $\chi_d(G) \leq 6 + \frac{3\Gamma+3}{\Gamma-2}((\Gamma-1)^{d-1} - 1)$ , where  $\Gamma = \max\{8, \Delta\}$ . Agnarsson and Halldórsson [1] proved for any fixed  $d$ ,  $G^d$  is  $O(\Delta^{\lceil d/2 \rceil})$ -colorable. Some authors studied  $\chi_d(G)$  for special graphs arisen from applications, such as square lattice [3] and hexagonal lattice [13]. Other related work could be found in [2] and [7]. It was proved by Sharp [16] that for fixed  $d \geq 2$  the distance coloring problem is polynomial time for  $k \leq \lceil 3d/2 \rceil$  and NP-hard for  $k > \lceil 3d/2 \rceil$ .

Some notations are introduced as follows. Let  $G$  be a graph. The degree of a vertex  $v$  is denoted by  $d_G(v)$  or  $d(v)$ . The maximum or minimum degree of  $G$  is denoted by  $\Delta(G)$  or  $\delta(G)$ , or simply  $\Delta$  or  $\delta$ . The distance between two vertices  $u$  and  $v$  is denoted by  $d_G(u, v)$  or simply  $d(u, v)$ . The diameter, girth and connectivity of  $G$  are denoted by  $\text{diam}(G)$ ,  $g(G)$ ,  $\kappa(G)$  respectively. For a vertex  $v \in V(G)$ , denote by  $N_d(v)$  ( $d \geq 1$ ) the set of vertices with distance  $d$  from  $v$  in  $G$ . Denote by  $G[U]$  the subgraph of  $G$  induced by the vertices of  $U \subseteq V(G)$ . If  $G$  has  $n$  vertices, the *adjacency matrix*  $G$ , denoted by  $A(G)$ , is defined as a symmetric  $(0, 1)$ -matrix of order  $n$ , where  $A(G)_{uv} = 1$  if and only if  $u$  is adjacent to  $v$ . A *Moore graph* is a regular graph of degree  $\Delta$  and diameter  $d$  whose number of vertices equals the upper bound  $1 + \Delta \sum_{i=1}^d (\Delta - 1)^{i-1}$ . An equivalent definition of a Moore graph is that it is a graph of diameter  $d$  with girth  $2d + 1$ . Moore graphs can be generalized to allow they have even girth, but are not used in this paper.

## 2 The upper bound of $\chi_d(G)$ in terms of maximum degree

When  $\Delta = 2$ , there exist only two connected graphs of order  $n$ : the cycle  $C_n$  and the path  $P_n$ . It is easy to get the result:

- (1)  $\chi_d(P_n) = \min\{n, d+1\}$ ;
- (2)  $\chi_d(C_n) = n$  if  $n \leq 2d+1$ .

For  $\chi_d(C_n)$  in the case of  $n \geq 2d+2$ , write  $n = m(d+1) + p$ , where  $0 \leq p < d+1$ , and  $p = (k-1)m + q$ , where  $0 \leq q < m$ . Then  $n = m(d+k) + q$ . So  $C_n^d$  is  $(d+k)$ -colorable if  $q = 0$  or is  $(d+k+1)$ -colorable otherwise.

- (3)  $\chi_d(C_n) = d+k$  if  $n \geq 2d+2$  and  $q = 0$ .
- (4)  $\chi_d(C_n) = d+k+1$  if  $n \geq 2d+2$  and  $q > 0$ .

The *inductiveness* of a graph  $G$ , also known as the *degeneracy*, the *coloring number*, and the *Szekeres-Wilf number*, denoted by  $\text{ind}(G)$  and is defined as  $\text{ind}(G) = \max_{H \subseteq G} \delta(H)$ , where  $H$  runs through all induced subgraphs of  $G$ . Inductiveness leads to an ordering of the vertices,  $\{v_1, v_2, \dots, v_n\}$ , such that  $d_{G[v_1, v_2, \dots, v_{i-1}]}(v_i) \leq \text{ind}(G)$ . It is known that  $\chi(G) \leq \text{ind}(G) + 1$ . In this section, we will find a spanning tree  $T$  of  $G$ , and use the postorder of  $T$  to order the vertices of  $G$ . According to this order, we upper-bound  $\text{ind}(G^d)$  and hence  $\chi(G^d)$  or  $\chi_d(G)$ . In the below, we improve the upper bound in (1.1) for  $\chi_d(G)$  under some conditions on the graph  $G$ .

**LEMMA 2.1** *Let  $G$  be a graph with  $\delta(G) \leq \Delta - 1$ . Then  $\chi_d(G) \leq \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2} - 1$ .*

**Proof.** Let  $d(v) = \delta(G)$  and  $T$  be a spanning tree of  $G$  with  $v =: v_n$  as its root. We order the vertices of  $G$  from  $v_1$  to  $v_n$  according to a postorder of  $T$  such that each  $v_i$  ( $1 \leq i \leq n-1$ ) is adjacent to some of  $v_j$  ( $j > i$ ). We will show  $\text{ind}(G) \leq \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2} - 2 =: M - 2$ , and then successively color  $v_1, v_2, \dots, v_n$  with at most  $M - 1$  colors.

For  $1 \leq i \leq n-2$ ,  $d_{G^d}(v_i) \leq M$ . Note that  $v_i$  has a neighbor  $v_j$  ( $j > i$ ). If  $j \leq n-1$ , then  $v_j$  has a neighbor  $v_k$  ( $k > j$ ), and hence  $d_{G^d[v_1, \dots, v_{i-1}]}(v_i) \leq M - 2$ . Otherwise,  $v_i$  is adjacent to  $v_n$ . As  $d(v_n) \leq \Delta - 1$ ,

$$d_{G^d}(v_i) \leq M - [1 + (\Delta - 1) + \dots + (\Delta - 1)^{d-2}] \leq M - 1,$$

which implies that  $d_{G^d[v_1, \dots, v_{i-1}]}(v_i) \leq (M - 1) - 1 = M - 2$ . Since  $v_{n-1}$  is adjacent to  $v_n$ , by the above discussion,  $d_{G^d}(v_{n-1}) \leq M - 1$ , and hence  $d_{G^d[v_1, \dots, v_{n-2}]}(v_{n-1}) \leq M - 2$ . For the last vertex  $v_n$ ,

$$d_{G^d}(v_n) \leq M - [1 + (\Delta - 1) + \dots + (\Delta - 1)^{d-1}] \leq M - \Delta \leq M - 2.$$

So we can color the graph  $G^d$  with at most  $M - 1$  colors. ■

**LEMMA 2.2** *Let  $G$  be a graph with  $g(G) \leq 2d$ . Then  $\chi_d(G) \leq \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2}$ .*

**Proof.** Let  $C$  be a shortest cycle of  $G$ ,  $u$  be an arbitrary vertex of  $C$ , and  $T$  be a spanning tree of  $G$  with  $u =: v_n$  as its root. Order the vertices of  $G$  from  $v_1$  to  $v_n$  according to a postorder of  $T$  such that each  $v_i$  ( $1 \leq i \leq n-1$ ) is adjacent to some of  $v_j$  ( $j > i$ ).

For  $1 \leq i \leq n-1$ ,  $d_{G^d}(v_i) \leq M := \Delta \frac{(\Delta-1)^d-1}{\Delta-2}$ . As  $v_i$  is adjacent to some  $v_j$  ( $j > i$ ),  $d_{G^d[v_1, \dots, v_{i-1}]}(v_i) \leq M-1$ . Since  $v_n$  is contained in the shortest cycle  $C$ , there exists some  $k \in \{2, 3, \dots, d\}$  such that  $N_k(v_n) \leq \Delta(\Delta-1)^{k-1}-1$ , which implies that  $d_{G^d}(v_n) \leq M-1$ . The result follows by a similar discussion as in Lemma 2.1.  $\blacksquare$

LEMMA 2.3 *Let  $G$  be a graph with  $g(G) \leq 2d-1$ . Then  $\chi_d(G) \leq \Delta \frac{(\Delta-1)^d-1}{\Delta-2} - 1$ .*

**Proof.** Similar to the proof of Lemma 2.2, we let  $C$  be a shortest cycle of  $G$ ,  $uv$  be an arbitrary edge of  $C$  and  $T$  be a spanning tree of  $G$  such that  $uv \in E(T)$ . Let the root of  $T$  be  $u =: v_n$ . Order the vertices of  $G$  from  $v_1$  to  $v_n$  such that each  $v_i$  ( $1 \leq i \leq n-1$ ) is adjacent to some of  $v_j$  ( $j > i$ ) and  $v =: v_{n-1}$ .

For  $1 \leq i \leq n-2$ ,  $v_i$  has a neighbor  $v_j$  ( $j > i$ ). If  $j \leq n-1$ , then  $v_j$  has a neighbor  $v_k$  ( $k > j$ ). If  $j = n$ , then  $v_j$  has a neighbor  $v_{n-1}$ . Hence  $d_{G^d[v_1, \dots, v_{i-1}]}(v_i) \leq M-2$  in either case.

For  $i = n-1$  or  $n$ ,  $v_i$  lies on the shortest cycle  $C$ , then there exists  $k \in \{2, 3, \dots, d\}$  such that  $N_k(v_i) \leq \Delta(\Delta-1)^{k-1}-2$ , which implies that  $d_{G^d}(v_i) \leq M-2$ . So we can color the graph  $G^d$  with at most  $M-1$  colors.  $\blacksquare$

THEOREM 2.4 *Let  $G$  be a graph. Then*

$$\chi_d(G) \leq \Delta \frac{(\Delta-1)^d-1}{\Delta-2} + 1,$$

*with equality if and only if  $G$  is  $\Delta$ -regular of order  $\Delta \frac{(\Delta-1)^d-1}{\Delta-2} + 1$ ,  $g(G) = 2d+1$ ,  $\text{diam}(G) = d$ , i.e.,  $G$  is a Moore graph.*

**Proof.** The upper bound follows from (1.1). We now discuss the equality case. For the necessity, assume  $\chi_d(G) = \Delta \frac{(\Delta-1)^d-1}{\Delta-2} + 1$ . By Lemmas 2.1 and 2.2,  $G$  is a  $\Delta$ -regular graph with  $g(G) \geq 2d+1$ . Since  $\chi_d(G) = \chi(G^d) = \Delta \frac{(\Delta-1)^d-1}{\Delta-2} + 1 \geq \Delta(G^d) + 1$ , by Brooks' Theorem,  $G^d$  is a complete graph. Then  $|V(G)| = \Delta(G^d) + 1$ ,  $\text{diam}(G) \leq d$ , and hence  $g(G) \leq 2d+1$ , which implies  $g(G) = 2d+1$  and  $\text{diam}(G) = d$ . So  $G$  is a Moore graph.

The sufficiency is easily obtained from the fact that  $G^d$  is a complete graphs of order  $\Delta \frac{(\Delta-1)^d-1}{\Delta-2} + 1$  as  $\text{diam}(G) = d$ .  $\blacksquare$

THEOREM 2.5 *Let  $G$  be a graph with  $g(G) \geq 2d+2$  ( $d \geq 2$ ) and  $\kappa(G) \geq 3$ . Then  $\chi_d(G) \leq \Delta \frac{(\Delta-1)^d-1}{\Delta-2}$ .*

**Proof.** Let  $C$  be a shortest cycle of  $G$  with length at least  $2d+2$ . Choose two vertices  $v_1, v_2$  on  $C$  with distance exactly  $d+1$ , and choose an arbitrary vertex  $v_n$  of  $C$  lying on the shortest path on  $C$  from  $v_1$  to  $v_2$ . As  $\kappa(G) \geq 3$ , the subgraph  $G \setminus \{v_1, v_2\}$  is connected. Let  $T$  be a spanning tree of  $G \setminus \{v_1, v_2\}$  with root  $v_n$ . Ordering the vertices of  $T$  according to a postorder of  $T$  from  $v_3$  to  $v_n$ , we can successively color  $v_1, v_2, \dots, v_n$  with at most  $\Delta \frac{(\Delta-1)^d-1}{\Delta-2} =: M$  colors. First we color  $v_1, v_2$  with color 1.

If  $1 \leq i \leq n-1$ , as  $v_i$  is adjacent to some vertex  $v_j$  ( $j > i$ ),  $d_{G^d[v_1, v_2, \dots, v_{i-1}]}(v_i) \leq M-1$ . Since  $v_n$  has two neighbors  $v_1, v_2$  colored with same color, we can color the neighbors of  $v_n$  in  $G^d$  with at most  $M-1$  colors, and use the remaining color to color  $v_n$ .  $\blacksquare$

**THEOREM 2.6** *Let  $G$  be a graph with  $g(G) \geq 2d+2$  and  $\kappa(G) \geq 5$ . Then  $\chi_d(G) \leq \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2} - 1$ .*

**Proof.** Let  $C$  be a shortest cycle with length at least  $2d+2$  in  $G$ . Choose a pair of vertices  $v_1, v_2$  on the cycle  $C$  such the distance between them is exactly  $d+1$ , choose a vertex  $v_3$  adjacent to  $v_2$ , and another vertex  $v_n$  adjacent to  $v_1$ , both on the shortest path on the cycle  $C$  from  $v_1$  to  $v_2$ . As  $\kappa(G) \geq 5$ , there exists a vertex  $v_{n-1}$  outside of  $C$ , joining  $v_n$  by an edge; there also exists a vertex  $v_4$  outside of  $C$  joining  $v_{n-1}$  by an edge; see Fig. 2.1 for the positions of these labeled vertices.

As  $\kappa(G) \geq 5$ , the subgraph  $G \setminus \{v_1, v_2, v_3, v_4\}$  is connected. Let  $T$  be a spanning tree of  $G \setminus \{v_1, v_2, v_3, v_4\}$  with root  $v_n$ , which contains the edge  $v_{n-1}v_n$ . Ordering the vertices of  $T$  according to a post order walk of  $T$  from  $v_5$  to  $v_n$ , we can successively color  $v_1, v_2, \dots, v_n$  with at most  $\Delta \frac{(\Delta-1)^{d-1}}{\Delta-2} - 1 =: M - 1$  colors. First we color  $v_1, v_2$  with color 1, and color  $v_3, v_4$  with color 2.

For all  $1 \leq i \leq n$ ,  $d_{G^d}(v_i) \leq M$ . If  $1 \leq i \leq n-2$ , as  $v_i$  has a neighbor  $v_j$  ( $j > i$ ), and  $v_j$  has a neighbor  $v_k$  ( $k > j$ ) if  $j \leq n-1$ ; if  $v_j = v_n$ , noting that  $v_n$  is adjacent to  $v_{n-1}$ , so  $v_j$  is also adjacent to  $v_{n-1}$  in  $G^d$ . Hence  $v_i$  has at most  $M-2$  neighbors in  $G^d[v_1, v_2, \dots, v_{i-1}]$  which can be colored by at most  $M-2$  colors, and then  $v_i$  can be colored by the remaining color.

For the vertex  $v_{n-1}$ , as it has two neighbors  $v_3, v_4$  (in  $G^d$ ) colored with same color and it is adjacent to  $v_n$ , we can color  $v_{n-1}$  and its neighbors in  $G^d[v_1, \dots, v_{n-2}]$  with at most  $M-1$  colors. Since  $v_n$  has four neighbors  $v_1, v_2, v_3, v_4$  (in  $G^d$ ) colored by 2 colors, we can color its neighbors (in  $G^d$ ) by at most  $M-2$  colors and then color  $v_n$  with the remaining color. ■

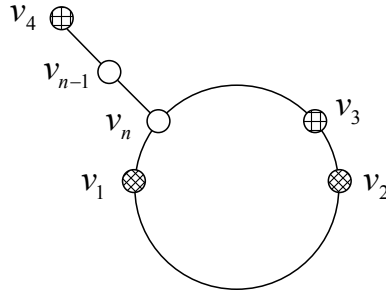


Fig. 2.1 An illustration of the proof of Theorem 2.6

**LEMMA 2.7** [15] *There is a  $\Delta_0$  such that if  $\Delta(G) \geq \Delta_0$  and  $\omega(G) \leq \Delta(G) - 1$  then  $\chi(G) \leq \Delta(G) - 1$ . In fact,  $\Delta_0 = 10^{14}$  will do.*

**THEOREM 2.8** *There exists a  $\Delta_0$  such that if  $\Delta(G) = \Delta$  is odd and  $\Delta \geq \Delta_0$  then  $\chi_d(G) \neq \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2}$ . In fact,  $\Delta_0 = (10^{14} + 1)^{\frac{1}{d}} + 1$  will do.*

**Proof.** Assume to the contrary, Let  $G$  be a graph with odd maximum degree  $\Delta \geq \Delta_0 = (10^{14} + 1)^{\frac{1}{d}} + 1$  and  $\chi_d(G) = \Delta \frac{(\Delta-1)^{d-1}}{\Delta-2} =: M$ . Then  $M > (\Delta - 1)^d - 1 \geq 10^{14}$ .

By Lemmas 2.3 and 2.1,  $G$  is  $\Delta$ -regular with girth  $g(G) \geq 2d$ . Clearly,  $M \geq \Delta(G^d)$ . By Brooks' theorem,  $\Delta(G^d)$  equals  $M$  or  $M-1$ . If  $\Delta(G^d) = M-1$ , then  $G^d$  is a complete graph of order  $M$  also by Brooks' theorem. Note that degree sum of the vertices of  $G$  is  $M \cdot \Delta$ , which is

an odd number; a contradiction. So  $\Delta(G^d) = M$ . By Lemma 2.7,  $G^d$  contains a  $M$ -clique  $K_M$ , that is,  $G$  has  $M$  vertices such that the distance between any two of them is at most  $d$ . For any  $v \in K_M$ , surely  $V(K_M) \setminus \{v\} \subseteq \cup_{j=1}^d N_j(v)$ .

First assume that for any  $v \in K_M$ ,  $V(K_M) \setminus \{v\} \subsetneq \cup_{j=1}^d N_j(v)$ . Let  $v \in V(K_M)$ . Then there is vertex  $w \in \cup_{j=1}^d N_j(v) \setminus V(K_M)$ , and hence  $d_{G^d}(v) \geq (M-1)+1 = \Delta(G^d)$ . So  $d_{G^d}(v) = \Delta(G^d)$  and  $\cup_{j=1}^d N_j(v) = (V(K_M) \setminus \{v\}) \cup \{w\}$ . Observe that  $N(w) \subseteq V(K_M)$ ; otherwise there exists a vertex  $u \in V(K_M)$ , adjacent to  $w$ , within distance 2 to a neighbor of  $w$  outside of  $K_M$ , so that  $d_{G^d}(u) \geq M+1 > \Delta(G^d)$ . Therefore  $d_{G[V(K_M) \cup \{w\}]}(w, v') \leq d+1$  for any vertex  $v' \in V(K_M)$ .

We assert that  $V(G) = V(K_M) \cup \{w\}$  and hence  $G$  is a Moore graph; but in this case  $\chi_d(G) = M+1$  by Theorem 2.4, a contradiction. If  $V(G) \neq V(K_M) \cup \{w\}$ , there exists a vertex  $\bar{u} \in V(K_M)$ , adjacent to some vertex outside of  $G[V(K_M) \cup \{w\}]$ . Clearly  $\bar{u} \neq w$  by the above discussion. Let  $P = \bar{u}w' \cdots w$  be a shortest path in  $G[V(K_M) \cup \{w\}]$  between  $\bar{u}$  and  $w$  with length  $l \leq d+1$ . Then  $d_{G^d}(w') \geq M+1$  if  $l > 1$  and  $d_{G^d}(\bar{u}) \geq M+1$  otherwise; a contradiction.

Next we assume there exists a vertex  $v \in K_M$  such that  $V(K_M) \setminus \{v\} = \cup_{j=1}^d N_j(v)$ . Then  $|\cup_{j=1}^d N_j(v)| = M-1$  and  $N_d(v) = \Delta(\Delta-1)^{d-1} - 1$  as  $g(G) \geq 2d$ , namely, there exists exactly one vertex  $w \in N_d(v)$  adjacent to two vertices in  $N_{d-1}(v)$ .

Note that any vertex in  $N_d(v)$  is within distance  $d$  to any one in  $N(v)$ . So, if counting the paths from a given vertex  $u \in N_d(v)$  to the vertices of  $N(v)$ , we find that  $u$  has  $\Delta-1$  neighbors in  $N_d(v)$  if  $u \notin N(w) \cup \{w\}$ ,  $\Delta-2$  neighbors in  $N_d(v)$  if  $u = w$ , and at least  $\Delta-2$  neighbors in  $N_d(v)$  if  $u \in N(w)$ . Hence the vertices with possible neighbors outside of  $V(K_M)$  belong to  $N(w)$ , and each such vertex has at most one neighbor outside of  $V(K_M)$ , which implies that at most  $\Delta-2$  edges between  $V(K_M)$  and the remaining vertices (if they exist).

We now assert that  $V(G) = V(K_M)$ . Otherwise, by the above discussion there exists a vertex  $\bar{u} \in N_d(v) \cap N(w)$  adjacent to a vertex  $w'$  outside of  $K_M$ . Then  $N(w') \subseteq N_d(v)$ , for otherwise  $d_{G^d}(\bar{u}) \geq M+1$ . But there are at most  $\Delta-2$  edges between  $V(K_M)$  and  $V(G) \setminus V(K_M)$ , so there are at most  $\Delta-2$  vertices in  $N_d(v)$  adjacent to  $w'$ . This contradicts the regularity of  $G$ . So  $V(G) = V(K_M)$ . Now the degree sum of the vertices of  $G$  is  $M \cdot \Delta$ , which is an odd number, a contradiction. The result follows by combining the above discussion.  $\blacksquare$

**Remark 1:** Let  $\mathcal{G}_\Delta$  be the class of graphs with maximum degree  $\Delta$ . It is known that for each graph  $G \in \mathcal{G}_\Delta$ ,

$$\chi_d(G) \in \left[ \Delta + 1, 1 + \frac{(\Delta-1)^d - 1}{\Delta-2} \right].$$

One would ask whether there exists a graph  $G \in \mathcal{G}_\Delta$  with  $\chi_d(G) = k$  for each integer  $k$  belonging to the above interval. We discuss this problem for  $d = 2$ .

It is known for a graph  $G \in \mathcal{G}_\Delta$ ,  $\chi_2(G) \in [\Delta+1, \Delta^2+1]$ , and  $\chi_2(G) = \Delta^2+1$  if and only if  $G$  is a Moore graph with diameter 2 and girth 5. By a little calculation,  $\chi_2(K_{k,\Delta}) = \Delta+k$  for  $1 \leq k \leq \Delta$ , and  $\chi_2(\hat{K}_{\Delta,\Delta}) = 2\Delta+1$ , where  $K_{m,n}$  denotes the complete graph with two parts having  $m$  and  $n$  vertices respectively, and  $\hat{K}_{\Delta,\Delta}$  is a partition of  $K_{\Delta,\Delta}$ . So we would ask for each  $k \in [2\Delta+2, \Delta^2]$  if there exist graphs  $G \in \mathcal{G}_\Delta$  with  $\chi_2(G) = k$ ?

When  $\Delta = 3$ , a graph obtained from Petersen graph by deleting an edge has the 2-distance

chromatic number 8. We do not find the graph of  $\mathcal{G}_3$  with 2-distance chromatic number being 9.

When  $\Delta = 4$ , a graph obtained from Petersen graph by adding a pendent edge to an arbitrary vertex has 2-distance chromatic number 10. Let  $G_1, G_2, G_3$  be the graphs in Fig. 2.2. We find that  $\chi_2(G_1 \setminus v_2) = 11$ ,  $\chi_2(G_1 + v_1v_2) = 12$ ,  $\chi_2(G_2) = 13$ , and  $\chi_2(G_3) = 14$ . The graph obtained from a Moore graph of order 17 with maximum degree 4 by deleting an edge has the 2-distance chromatic number 15. We could not find the graph of  $\mathcal{G}_4$  with 2-distance chromatic number being 16.

So we conjecture that there does not exist any graph with maximum  $\Delta$  and the 2-distance chromatic number  $\Delta^2$ .

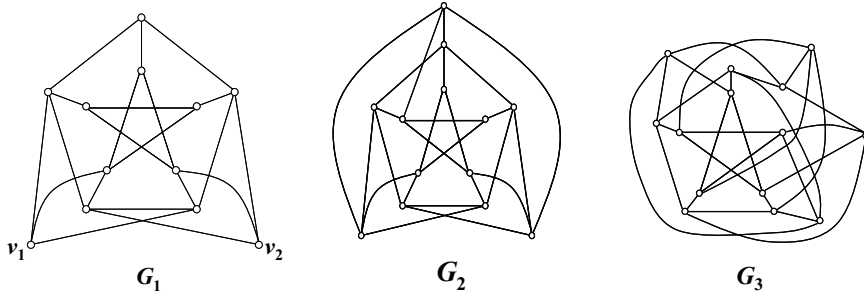


Fig. 2.2. The graphs  $G_1, G_2, G_3$

### 3 Upper bound of $\chi_d(G)$ in terms of spectral radius

Let  $A$  be a (entrywise) nonnegative matrix. By Perron-Frobenius theorem, the eigenvector of  $A$  corresponding the largest eigenvalue or spectral radius is (entrywise) nonnegative; this vector is also called the *Perron vector* of  $A$ . Denote by  $A \leq B$  for two matrices of the same size if each entry of  $A$  is less than or equal to the entry of  $B$  in same position. Let  $G$  be a graph. Observe that  $A(G)_{uv}^k$  is the number of walks in  $G$  from  $u$  to  $v$  with length  $k$ . So we have

$$A(G^d) \leq A(G) + A(G)^2 + \cdots + A(G)^d. \quad (3.1)$$

Denote by  $\lambda_1(G)$  (respectively,  $\lambda_1(A)$ ) or simply  $\lambda_1$  the spectral radius or the largest eigenvalue of  $A(G)$  (respectively, a square matrix  $A$ ).

LEMMA 3.1 [20] *Let  $G$  be a graph. Then  $\chi(G) \leq \lambda_1(G) + 1$ , with equality if and only if  $G$  is a complete graph or odd cycle.*

By Lemma 3.1 and (3.1), we easily get

$$\chi_d(G) = \chi(G^d) \leq \lambda_1 + \lambda_1^2 + \cdots + \lambda_1^d + 1 = \frac{\lambda_1^{d+1} - 1}{\lambda_1 - 1}. \quad (3.2)$$

However, the upper bound in (3.2) is too large. We improve it as follows, before we prove some basic facts.

LEMMA 3.2 *Let  $A, B$  be two nonzero square nonnegative matrix of same order. Then*

$$\lambda_1(AB) \leq \lambda_1(A)\lambda_1(B),$$

*with equality if and only if  $A, B, AB$  share a common Perron vector.*

**Proof.** There exists a unit Perron vector  $x$  such that  $\lambda_1(AB) = x^T ABx$ . So, by Cauchy-Schwarz inequality

$$\lambda_1(AB) = x^T ABx \leq \|Ax\| \cdot \|Bx\| = (x^T A^2 x)^{1/2} \cdot (x^T B^2 x)^{1/2} \leq \lambda_1(A)\lambda_1(B),$$

with equality if and only if  $Ax = \lambda_1(A)x, Bx = \lambda_1(B)x$ , i.e.  $A, B, AB$  share a common Perron vector. ■

LEMMA 3.3 *Suppose that  $G$  is a connected graph on at least 3 vertices.*

- (1)  $A(G^2) \leq A(G) + A(G)^2 - D(G)$  with equality if and only if  $G$  contains no  $C_3$  or  $C_4$ .
- (2) If  $d \geq 3$  and further  $G$  has no pendant edges, then  $A(G^d) \leq A(G^{d-1}) \cdot A(G)$ .

**Proof.** (1) We know that  $A(G^2) \leq A(G) + A(G)^2$ . But  $A(G)^2$  contains nonzero diagonal entries. In fact,  $A(G)_{vv}^2 = d(v)$ . So  $A(G^2) \leq A(G) + A(G)^2 - D(G)$ . Surely both sides have zero trace. If the equality holds, then  $A(G)_{uv} = 1$  implies  $A(G)_{uv}^2 = 0$ , that is,  $G$  contains no  $C_3$ . Furthermore, if  $A(G)_{uv}^2 \neq 0$ , then  $A(G)_{uv}^2 = 1$ , i.e. there is exactly one walk of length 2 from  $u$  to  $v$ . Hence  $G$  contains no  $C_4$ .

Conversely, if  $G$  contains no  $C_3$  or  $C_4$ , then for each edge  $uv$  of  $G^2$  (equivalently  $A(G^2)_{uv} = 1$ ),  $u$  and  $v$  are joined by an edge or by exactly one walk of length 2 (but cannot happen at the same time). So  $[A(G) + A(G)^2 - D(G)]_{uv} = 1$ . If  $uv$  ( $u \neq v$ ) is not an edge of  $G^2$  (equivalently  $A(G^2)_{uv} = 0$ ), then  $d(u, v) \geq 3$ , and hence  $[A(G) + A(G)^2 - D(G)]_{uv} = 0$ .

(2) For each edge  $uv$  in  $G^d$ , there is a path  $W$  of length  $l \leq d$  connecting  $u$  and  $v$ . If  $l \geq 2$ , writing  $W = u \cdots wv$ , then  $uw$  is an edge of  $G^{d-1}$  and  $wv$  is an edge of  $G$ , and hence  $A(G^d)_{uv} \leq [A(G^{d-1}) \cdot A]_{uv}$ . Otherwise,  $W = uv$ , by the condition on  $G$ ,  $v$  has a neighbor  $w$ , then  $G^{d-1}$  contains an edge  $uw$  and  $G$  contains an edge  $wv$ . The result follows by a similar discussion. ■

LEMMA 3.4 *Suppose a graph  $G$  is connected on at least 3 vertices.*

- (1)  $\lambda_1(G^2) \leq \lambda_1(G)^2$  with equality if and only if  $G$  is regular and contains no  $C_3$  or  $C_4$ .
- (2) If  $d \geq 3$  and further  $G$  has no pendant edges,  $\lambda_1(G^d) < \lambda_1(G)^d$ .

**Proof.** (1) By Lemma 3.3(1) and the theory of nonnegative matrices

$$\lambda_1(A(G^2)) \leq \lambda_1(A(G)^2 + A(G) - D(G)), \tag{3.3}$$

with equality if and only if  $A(G^2) = A(G) + A(G)^2 - D(G)$  as  $G^2$  is connected or  $A(G^2)$  is irreducible. Hence, also by Lemma 3.3(1), the equality in (3.3) holds if and only if  $G$  contains no  $C_3$  or  $C_4$ .



Noting that  $D(G) - A(G)$  (also called the Laplacian matrix of  $G$ ) is singular and positive semi-definite, hence by Wely's inequality

$$\lambda_1(A(G)^2 + A(G) - D(G)) \leq \lambda_1(A(G)^2) + \lambda_1(A(G) - D(G)) = \lambda_1(G)^2, \quad (3.4)$$

with equality if and only if  $A(G)^2 + A(G) - D(G)$ ,  $A(G)^2$  (or  $A(G)$ ), and  $A(G) - D(G)$  share a common eigenvector corresponding to their largest eigenvalues. But  $A(G) - D(G)$  has a simple largest eigenvalue 0 with the all-one vector as the corresponding eigenvector since  $G$  is connected. So the equality in (3.4) holds if and only if  $G$  is regular.

(2) By Lemma 3.3(2),  $A(G^d) \leq A(G^2) \cdot A(G)^{d-2}$ . So, by Lemma 3.2 and the result (1),

$$\lambda_1(G^d) = \lambda_1(A(G^d)) \leq \lambda_1(A(G^2) \cdot A(G)^{d-2}) \leq \lambda_1(A(G^2)) \cdot \lambda_1(A(G)^{d-2}) \leq \lambda_1(G)^d.$$

■

**THEOREM 3.5** *Let  $G$  be a connected graph on at least 3 vertices. Then  $\chi_2(G) \leq \lambda_1(G)^2 + 1$ , with equality holds if and only if  $G$  is a Moore graph with diameter 2 and girth 5. Furthermore, if  $G$  contains no pendent edges, then  $\chi_d(G) \leq \lambda_1(G)^d + 1$ .*

**Proof.** By Lemmas 3.1 and 3.4,

$$\chi_d(G) = \chi(G^d) \leq \lambda_1(G^d) + 1 \leq \lambda_1(G)^d + 1.$$

In the case of  $d = 2$ , we have

$$\chi_2(G) \leq \lambda_1(G^2) + 1 \leq \lambda_1(G)^2 + 1. \quad (3.5)$$

If (3.5) holds equalities, then from the first equality,  $G^2$  is complete which implies that  $\text{diam}(G) \leq 2$ ; and from the second equality,  $G$  is regular and contains no  $C_3$  or  $C_4$ . So,  $\text{diam}(G) = 2$ ,  $g(G) = 5$ , i.e.  $G$  is a Moore graph with diameter 2 and girth 5. For the sufficiency, if  $G$  is a Moore graph with degree  $\Delta$ , by Theorem 2.4,  $\chi_2(G) = \Delta^2 + 1$ . Observing  $\lambda_1(G) = \Delta$ , we get the equality. ■

**Remark 2:** If  $G$  is  $\Delta$ -regular, then  $\chi_2(G) \leq \Delta^2 + 1$ , which is consistent with the bound  $\lambda_1(G)^2 + 1$  since  $\lambda_1(G) = \Delta$  in this case. Otherwise, by Lemma 2.2,  $\chi_2(G) \leq \Delta^2 - 1$ . In general,  $\lambda_1(G) < \Delta$  when  $G$  is irregular with maximum degree  $\Delta$ . A special example is that  $G$  is a star on  $n$  vertices. Then Lemma 2.2 gives  $\chi_2(G) \leq n^2 + 2n$ , while Theorem 3.2 gives  $\chi_2(G) \leq n$ , that latter of which is an equality as the  $G^2$  is complete.

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